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EXAMINATION OF THE SOLUTIONS OF THE NAVIER-STOKES EQUATIONS FOR A CLASS OF THREE-DIMENSIONAL VORTICES

Part III: Temperature Distributions for Steady Motion

by

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ABSTRACT

The temperature distributions that occur inside a steadily rotated cylindrical vortex tube with porous walls having steady radial inflow of an essentially incompressible fluid are obtained. The nature of the distributions for both liquids and gases is discussed. The results are compared with Rott's results for Burgers' unbounded vortex.

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1. Introduction

In reference 1 (hereafter referred to as Part I) a class of solutions of the Navier-Stokes equations for the steady flow of an incompressible viscous medium are discussed in some detail. These flows are such that in terms of a cylindrical coordinate system (r, θ, z), the respective velocity components (u, v, w) are of the form

$$\begin{aligned} u &= u(r) \\ v &= v(r) \\ w &= z\overline{w}(r) \end{aligned} \tag{1.1}$$

and the pressure is of the form

$$p = \frac{1}{2}\rho\overline{C}z^2 + p(r) \tag{1.2}$$

where ρ is the density and $\overline{C}$ is a constant. It has been pointed out by Donaldson (reference 2) that, for such "essentially incompressible" flows, the energy equation can be easily solved for temperature distributions of the form

$$T = T_0(r) + z^2T_2(r) \tag{1.3}$$

In the present part (Part III), we give the results of a brief study of the nature of such temperature distributions, partly for the sake of completeness and partly because of a general interest in temperature distributions within vortex flows. The results that are obtained are closely related to the work of Rott (reference 3) who studied the temperature distributions in a free or unbounded vortex of the form $u = -ar$, $v = v(r)$, and $w = 2az$. The present study shows that the temperature distributions in confined vortices are very similar to those of free vortices.

Attention is confined to vortices of the family designated I-1 in Part I, i.e. single-cell vortices in which the flow spirals inward from a porous wall at $r = R$ and out along the axis. One boundary condition on the temperature is taken to be

$$T(R, z) = T_w = \text{constant} \tag{1.4}$$

The only other condition needed is that the temperature remain finite on the axis.

2. The Energy Equation

In Part I (pp. 16-22) the energy equation is discussed in detail. Two forms, one for gases and one for liquids, which are consistent with the constant density solutions of the continuity and momentum equations given in Part I are deduced. The equation for gases, suitable for cases in which the Mach number associated with the tangential velocity, v, is appreciable while that associated with u or w is negligible, is,* subject to (1.1),

$$(2.1)\quad \rho c_p\left(u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z}\right) - \frac{\rho u v^2}{r} = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi''$$

where the dissipation is

$$(2.2)\qquad \Phi'' = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)^2$$

As usual c_p denotes the specific heat at constant pressure and k the thermal conductivity, both assumed constant.

For liquids it is shown in Part I that the energy equation, again subject to (1.1), is

$$(2.3)\qquad \rho c_p\left(u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \Phi'$$

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*In Part I, (1.5.23) is written incorrectly. The terms

$$+\frac{\partial p}{\partial t} + \frac{\rho u v^2}{r}$$

should read

$$-\frac{\partial p}{\partial t} - \frac{\rho u v^2}{r}$$

where the dissipation is

$$(2.4)\quad \Phi' = 2\mu\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{u}{r}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \mu\left[\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2\right]$$

As pointed out by Goldstein (reference 4), there has been some confusion in the literature about the correct form of the energy equation for the case of "nearly incompressible" fluids. The forms given above are believed to be the most useful and are consistent with both the recent work of Rott (reference 3) and the suggestions of Goldstein.

The necessity of assuming that the Mach number associated with u or w is negligible leads to a very simple expression for the dissipation for gases, Φ'', which makes the solution relatively easy. The more complicated form for liquids, Φ', makes the solution more involved but gives a greater richness of information.

In (2.1) through (2.4), u, v, and w are known functions that are determined by the methods outlined in Part I. It is desirable at this point to review briefly the way the solutions of family I-1 are obtained. First, the following non-dimensional variables are defined:

$$(2.5)\qquad X = cr^2/R^2$$

$$(2.6)\qquad F(X) = -\frac{1}{2}ur/\nu$$

$$(2.7)\qquad H(X) = (vr/VR)H(c)$$

where V is the value of v at the wall, $\nu = \mu/\rho$, and

$$(2.8)\qquad c^2 = -\overline{C}R^4/(16\nu^2)$$

Here $\overline{C}$ is the pressure-gradient constant appearing in (1.2); it is always negative for family I-1, so c is real. The equation of continuity now shows that

$$w = z\overline{w} = 4c(\nu/R)(z/R)F' \tag{2.9}$$

where the prime denotes differentiation with respect to X.

As shown in Part I, pp. 34-38, F and H are governed by the equations

$$(XF'')' + FF'' - F'^2 + 1 = 0 \tag{2.10}$$

$$XH'' + FH' = 0 \tag{2.11}$$

These equations are integrated numerically with the initial conditions given by

$$F(0) = 0, \qquad F'(0) = A, \qquad F''(0) = A^2 - 1 \tag{2.12}$$

$$H(0) = 0, \qquad H'(0) = B \tag{2.13}$$

where the constant A is (for family I-1) between 0 and 1, and the constant B is arbitrary. Since it is specified as a boundary condition that $w = 0$ at the wall $r = R$, it is seen from (2.5) and (2.9) that $F'(c) = 0$. Hence for each A in (2.12) integration of (2.10) continues to the first (for family I-1) zero of F'. The value of X at which it occurs determines the value of c corresponding to the particular A. Integration of (2.11) can be performed simultaneously or separately. The other characteristics of the flow are determined from the knowledge of the functions F and H.

Accordingly (2.1) and (2.3) are rewritten in terms of the dimensionless variables and at the same time the assumption (1.3) is introduced in the modified form:

$$T = T_0(X) + \frac{z^2}{R^2} T_2(X) \tag{2.14}$$

It is found that each term is either independent of z or else multiplied by $(z/R)^2$. Since the equality must hold for all values of z, two equations are deduced from each of (2.1) and (2.3). For gases they are

$$(XT_2')' + Pr(FT_2' - 2F'T_2) = 0 \tag{2.15}$$

$$\begin{aligned}(XT_0')' + PrFT_0' &= - \frac{1}{2c} T_2 \\ &+ Pr \frac{V^2 c}{c_p H^2(c)}\left[\frac{F}{2}\frac{H^2}{X^2} - \left(H' - \frac{H}{X}\right)^2\right]\end{aligned} \tag{2.16}$$

while for liquids they are

$$(XT_2')' + Pr(FT_2' - 2F'T_2) = -16 \; Pr \frac{\nu^2 c^2}{c_p R^2} XF''^2 \tag{2.17}$$

$$\begin{aligned}(XT_0')' + PrFT_0' = - \frac{1}{2c} T_2 - 16 \; Pr \frac{\nu^2 c}{c_p R^2}\left[F'^2 - F'\frac{F}{2X} + \left(\frac{F}{2X}\right)^2\right] \\ - Pr \frac{V^2 c}{c_p H^2(c)}\left(H' - \frac{H}{X}\right)^2\end{aligned} \tag{2.18}$$

In these equations Pr is the Prandtl number, $\mu c_p/k$.

In both sets the boundary conditions are that T_0 and T_2 be finite at the origin, i.e.

$$T_0(0) < \infty, \qquad T_2(0) < \infty \tag{2.19}$$

and, by (1.4) and (2.5),

$$T_2(c) = 0, \qquad T_0(c) = T_w \tag{2.20}$$

3. Solution in the Case of Gases

We define the function $G(X)$ to be the solution (determined numerically) of the differential equation

$$(3.1) \qquad (XG')' + \mathrm{Pr} F G' = \mathrm{Pr}\left[\frac{F}{2}\frac{H^2}{X^2} - \left(H' - \frac{H}{X}\right)^2\right]$$

with the initial conditions

$$(3.2) \qquad G(0) = 0, \qquad G'(0) = 0$$

Further, we define

$$(3.3) \qquad \Lambda_G = \frac{c}{H^2(c)}\left[G(x) - G(c)\right]$$

Then it can be verified that the set

$$(3.4) \qquad T_2 = 0$$

$$(3.5) \qquad T_0 = T_W + \frac{V^2}{c_p}\Lambda_G$$

is a solution of (2.15) and (2.16) which satisfies the boundary conditions (2.19) and (2.20).

The computations have been made for a range of the Reynolds number $N_U = UR/\nu$, where $U = u(R) < 0$ (for family I-1), and for two values of the Prandtl number, $\mathrm{Pr} = 1/2$ and $\mathrm{Pr} = 1$. Typical distributions of

$$(3.6) \qquad \frac{c_p(T - T_W)}{V^2} = \Lambda_G$$

are shown in Figures 3.1, 3.2, and 3.3. Figure 3.4 shows the variation of the minimum value

$$\Lambda_{G_{min}} = \Lambda_G(0)$$

with Reynolds number. These results are discussed in Section 5.

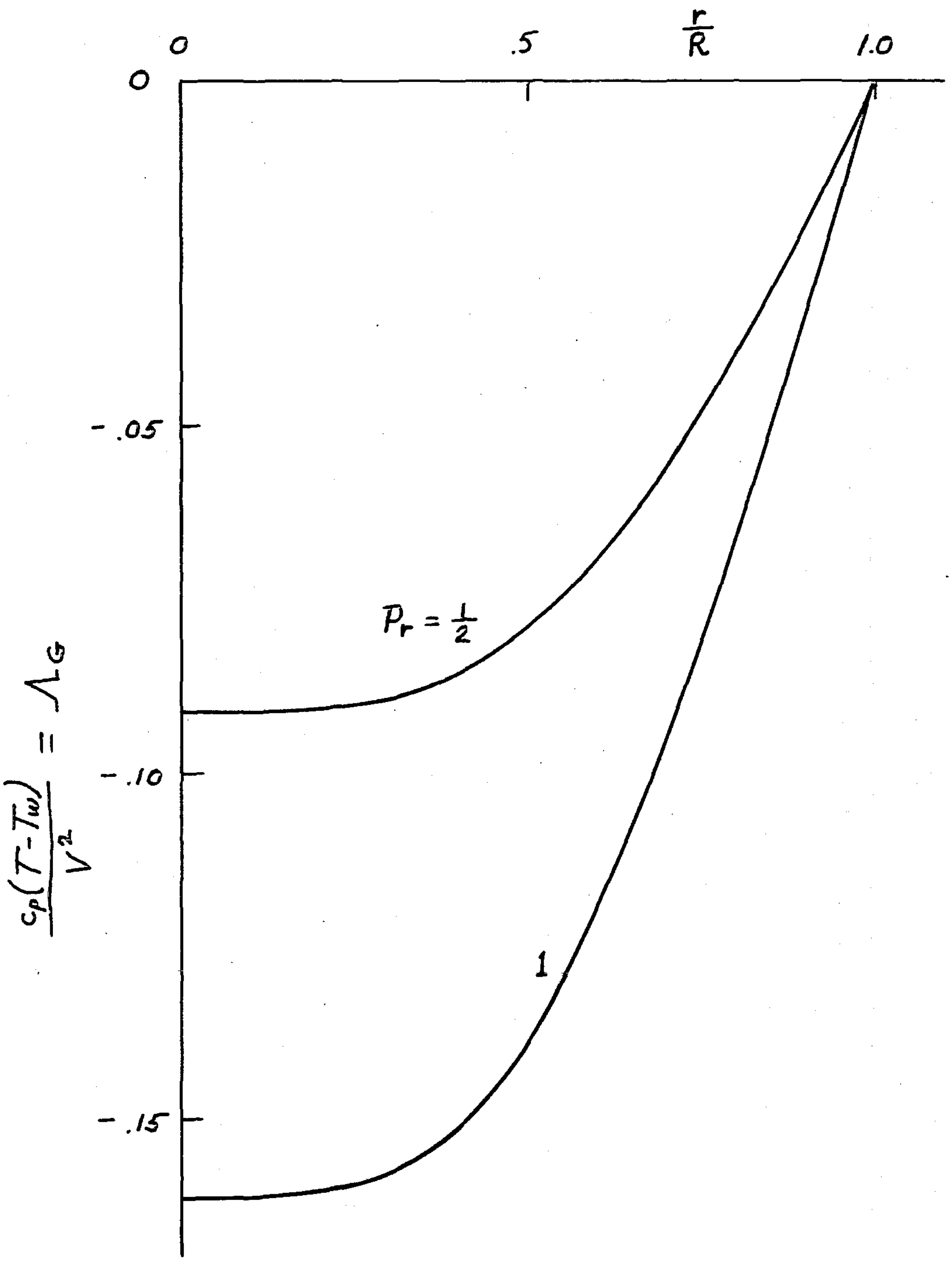

Fig. 3.1 Temperature distribution for gases at $N_U = -1.48$.

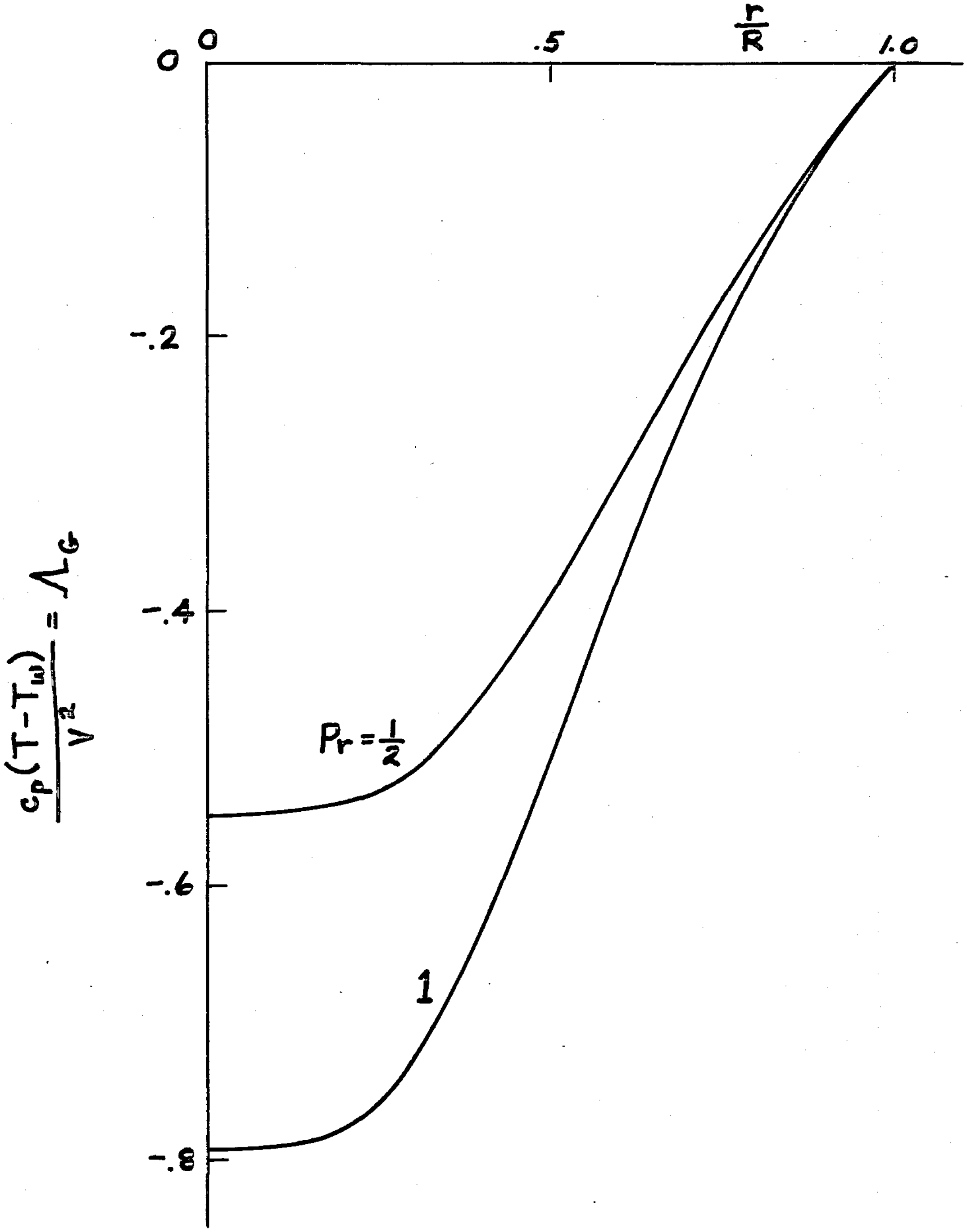

Fig. 3.2 Temperature distribution for gases at $N_U = -5.38$.

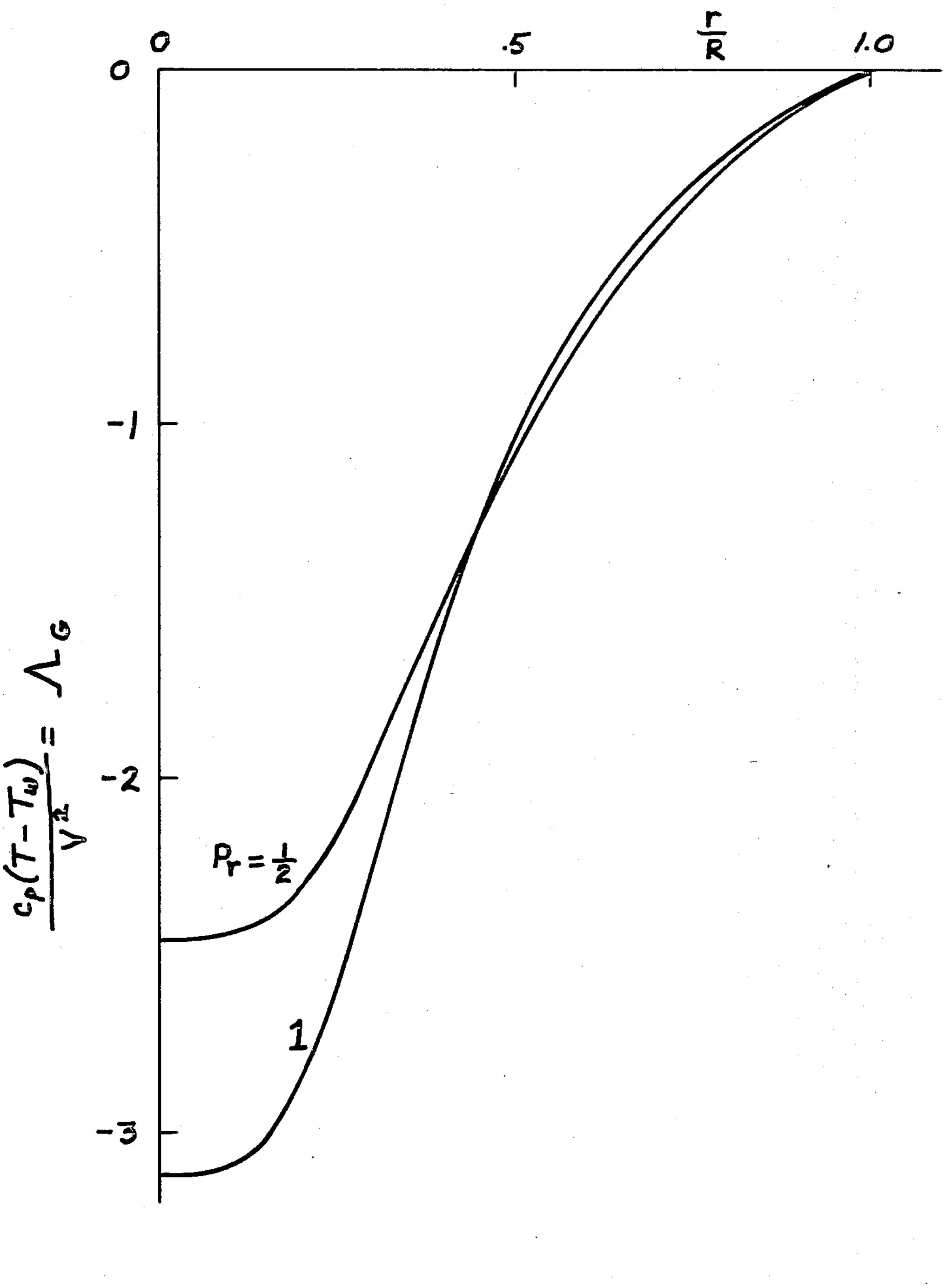

Fig. 3.3 Temperature distribution for gases at $N_U = -16.0$.

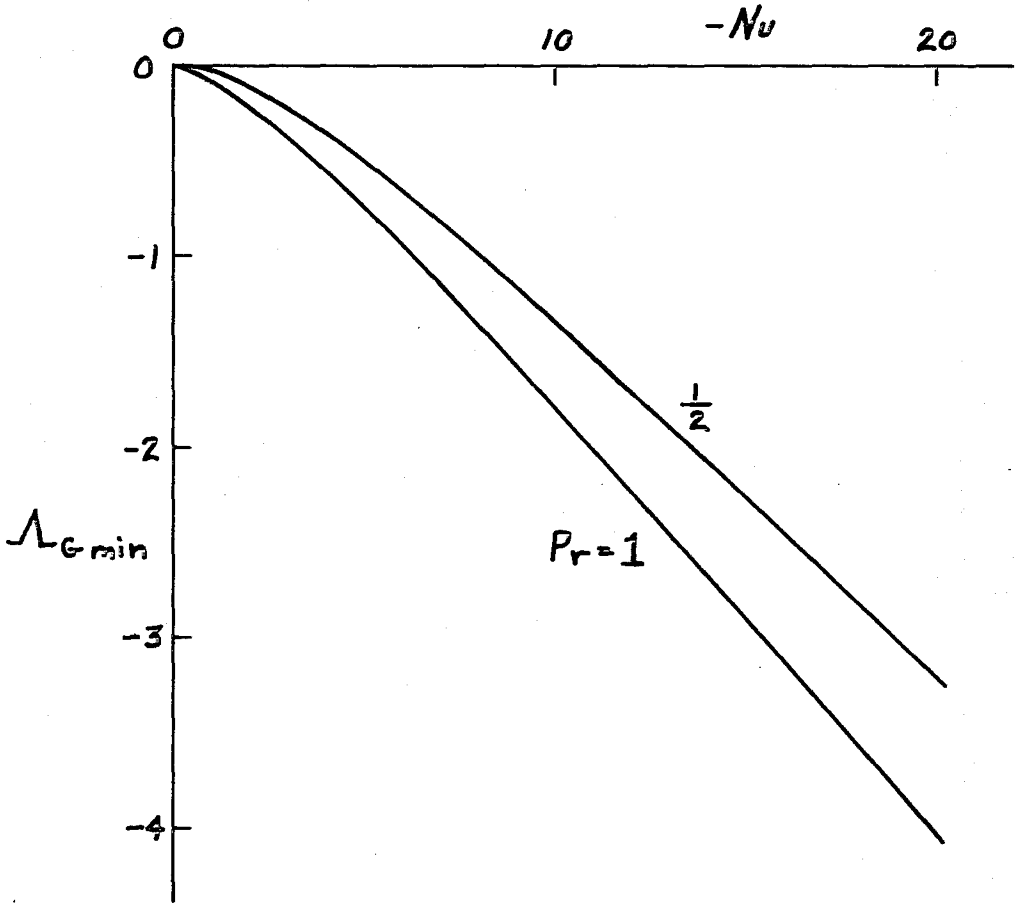

Fig. 3.4 Minimum of Λ_G as a function of Reynolds number.

4. Solution in the Case of Liquids

The technique of solution in the case of liquids is similar to that just outlined for gases, but considerably more involved. We define five functions, A_2, B_2, A_F, A_T, and A_H, through their differential equations and initial conditions as follows

$$(4.1)\quad (XA_2')' + Pr(FA_2' - 2F'A_2) = -PrXF''^2$$

$$(4.2)\quad A_2(0) = C, \qquad A_2'(0) = 2PrAC$$

$$(4.3)\quad (XB_2')' + Pr(FB_2' - 2F'B_2) = 0$$

$$(4.4)\quad B_2(0) = D, \qquad B_2'(0) = 2PrAD$$

$$(4.5)\quad (XA_F')' + PrFA_F' = -Pr\left[F'^2 - F'\frac{F}{2X} + \left(\frac{F}{2X}\right)^2\right] - \frac{1}{2}A_2$$

$$(4.6)\quad A_F(0) = 0, \qquad A_F'(0) = -\frac{3}{4}PrA^2 - \frac{1}{2}C$$

$$(4.7)\quad (XA_T')' + PrFA_T' = B_2$$

$$(4.8)\quad A_T(0) = 0, \qquad A_T'(0) = D$$

$$(4.9)\quad (XA_H')' + PrFA_H' = -Pr\left(H' - \frac{H}{X}\right)^2$$

$$(4.10)\quad A_H(0) = 0, \qquad A_H'(0) = 0$$

In these equations C and D, like B in (2.13), may be given any value which is convenient in the numerical integration. Further, we define

$$(4.11) \qquad \Lambda_0 = \frac{16\ c}{N_U^2}\left\{A_F - A_F(c) + \frac{K}{2}\left[A_T - A_T(c)\right]\right\}$$

$$(4.12) \qquad \Lambda_2 = \frac{16\ c^2}{N_U^2}\left(A_2 - KB_2\right)$$

and

$$(4.13) \qquad \Lambda_V = \frac{c}{H^2(c)}\left[A_H - A_H(c)\right]$$

where

$$(4.14) \qquad K = \frac{A_2(c)}{B_2(c)}$$

Then it can be verified that the set

$$(4.15) \qquad T_2 = \frac{U^2}{c_p}\Lambda_2$$

$$(4.16) \qquad T_0 = T_w + \frac{U^2}{c_p}\Lambda_0 + \frac{V^2}{c_p}\Lambda_V$$

is a solution of (2.17) and (2.18) which satisfies the boundary conditions (2.19) and (2.20).

Again the computations have been carried out for a range of the Reynolds number N_U and for two values of the Prandtl number, $Pr = 1/2$ and $Pr = 1$. Typical distributions of Λ_0, Λ_2, and Λ_V are shown in Figures 4.1, 4.2, and 4.3. Figure 4.4 shows the variation of the maxima of the same quantities with Reynolds number.

From (2.14), (4.15), and (4.16) we find

$$(4.17) \qquad c_p(T - T_w) = U^2\left[\Lambda_0 + \left(\frac{z}{R}\right)^2\Lambda_2\right] + V^2\Lambda_V$$

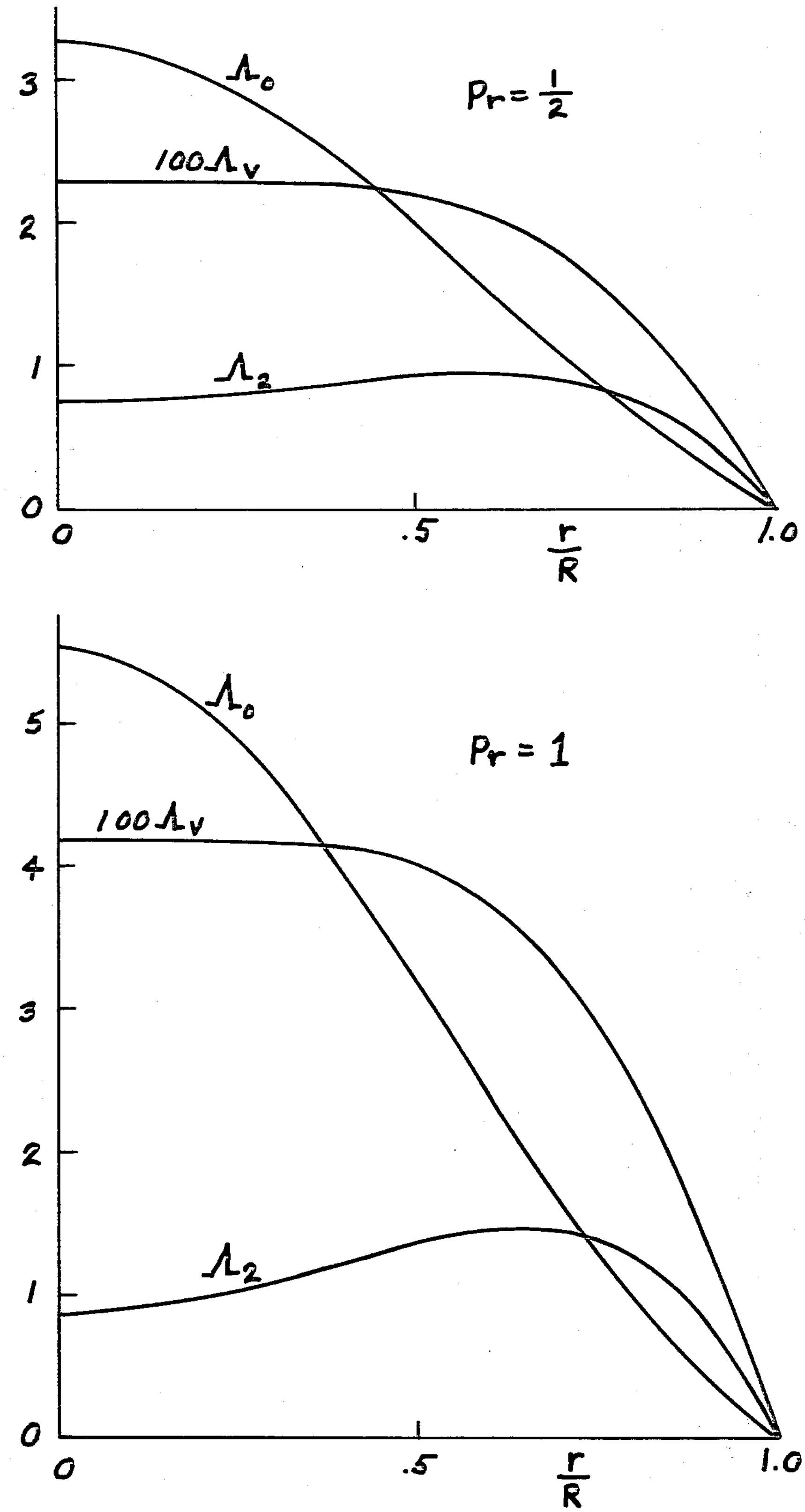

Fig. 4.1 The Λ functions for liquids at $N_U = -1.48$.

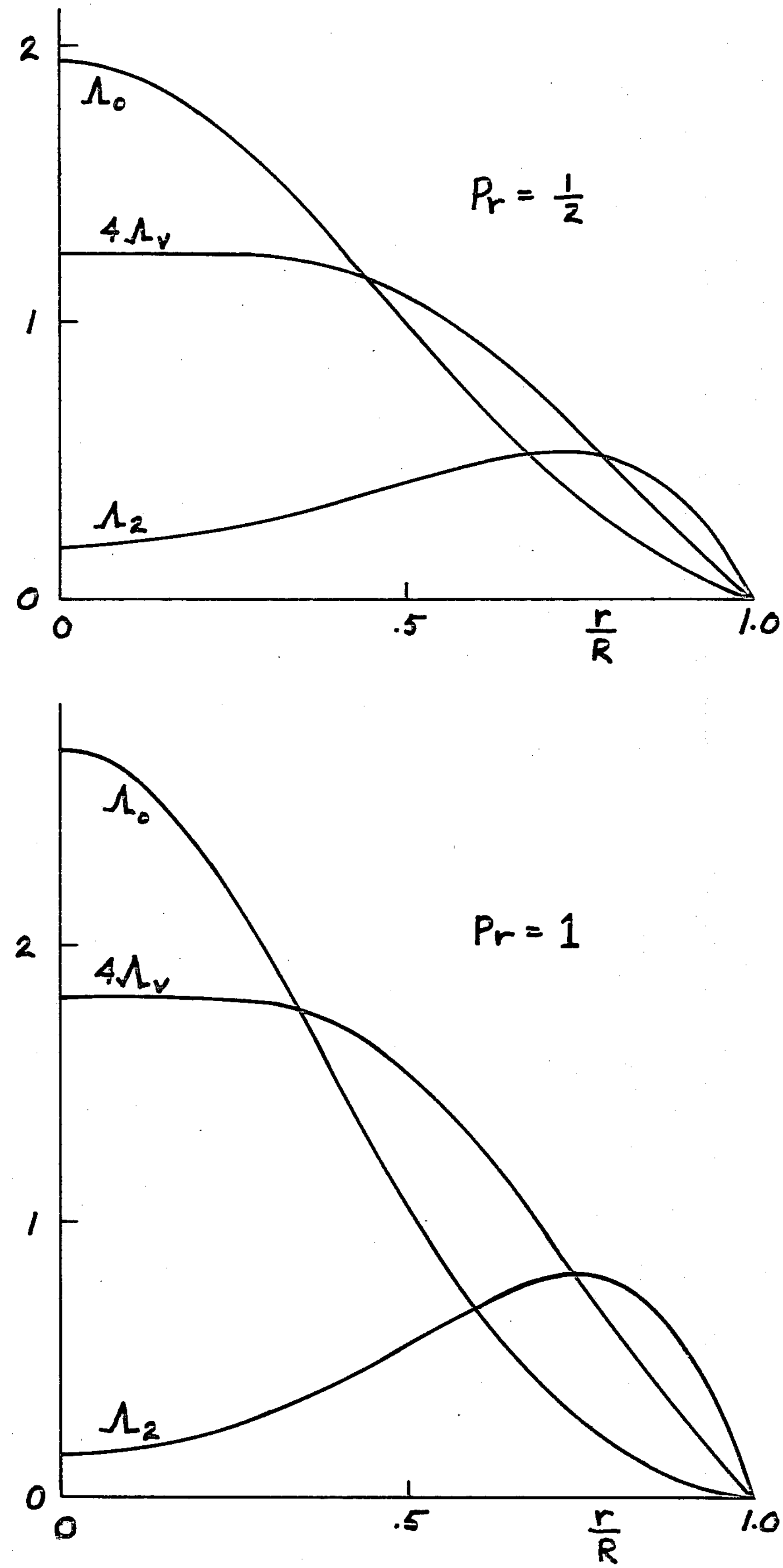

Fig. 4.2 The Λ functions for liquids at $N_U = -5.38$.

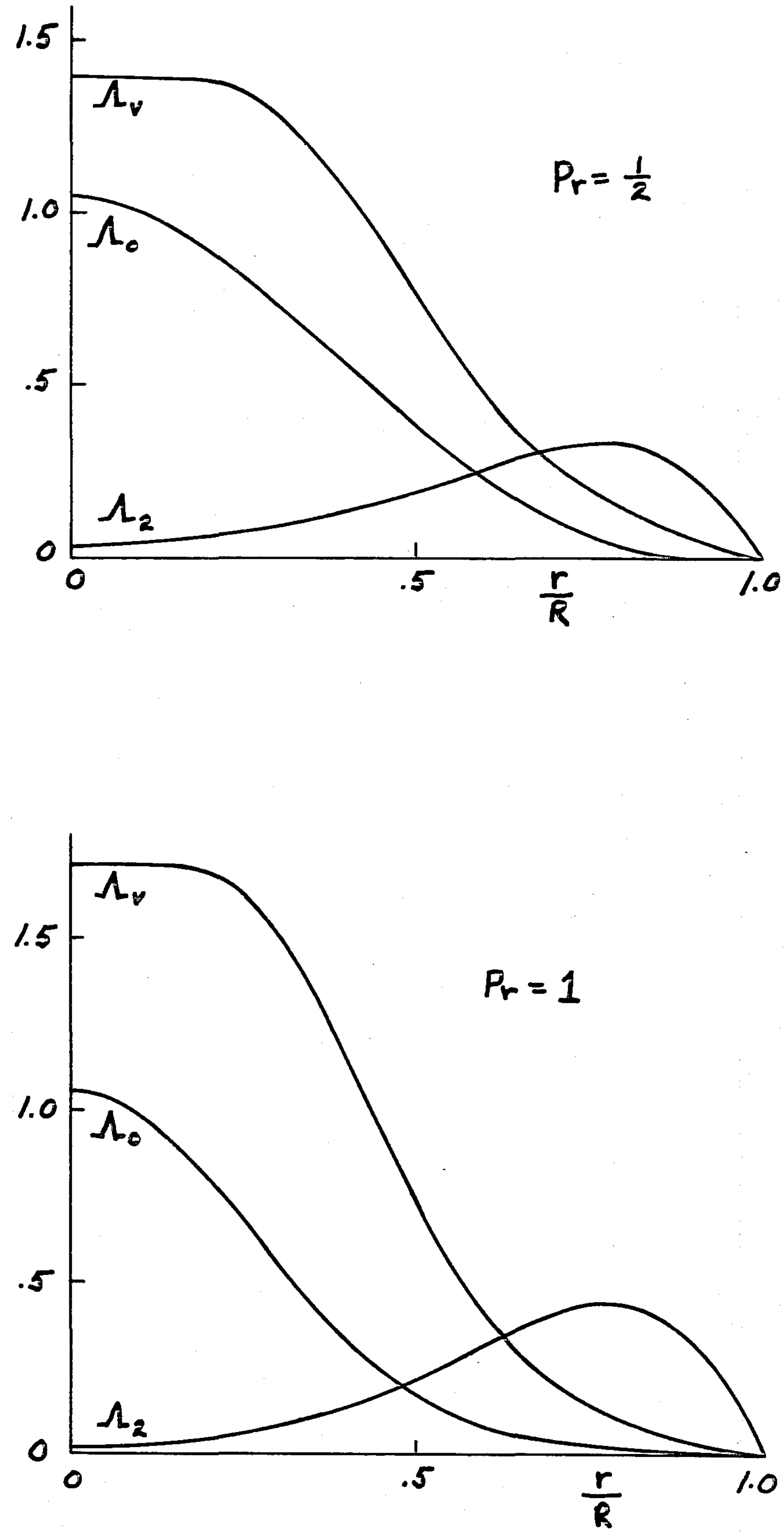

Fig. 4.3 The Λ functions for liquids at $N_U = -16.0$.

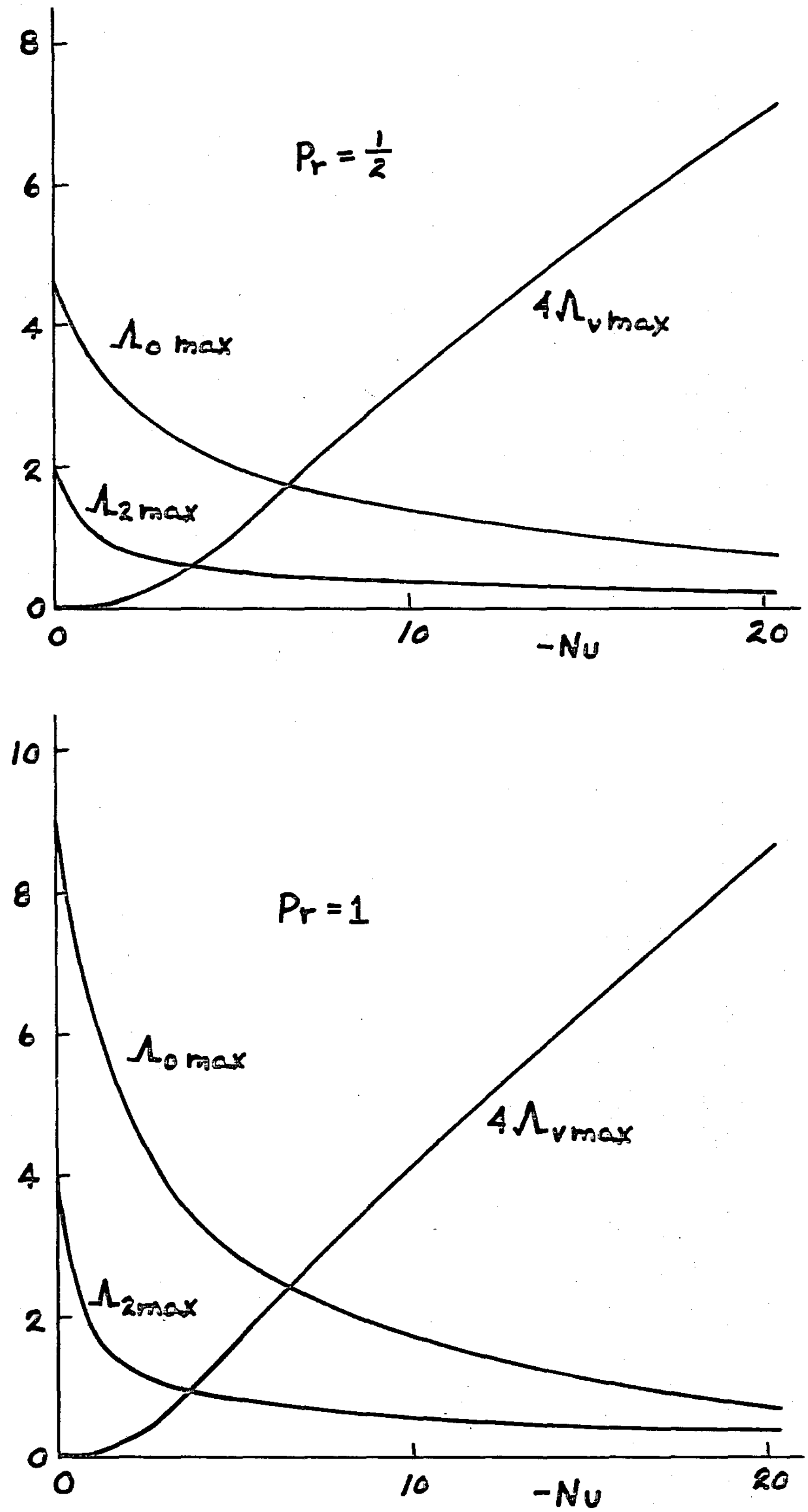

Fig. 4.4 Maxima of the Λ functions for liquids as a function of Reynolds number.

Thus for small z/R, Λ_V may be interpreted as

$$(4.18) \qquad \Lambda_V = \frac{c_p(T - T_w)}{V^2}$$

if V is sufficiently large compared to U while Λ_0 may be interpreted as

$$(4.19) \qquad \Lambda_0 = \frac{c_p(T - T_w)}{U^2}$$

if U is sufficiently large compared to V. But note from Figures 4.1 and 4.4 that when $|N_U|$ is small V must be very much larger indeed than U for the Λ_V term to dominate, and that when $|N_U|$ is large, the opposite is true. Of course, for z/R sufficiently large, Λ_2 may be interpreted as

$$(4.20) \qquad \Lambda_2 = \left(\frac{R}{z}\right)^2 \frac{c_p(T - T_w)}{U^2}$$

These results are discussed further in the following section.

5. Discussion

From the physical point of view, the most interesting aspect of the results is probably the disparity in the behavior of liquids and gases. That is, liquids are heated as they move toward the center but gases are cooled. This phenomenon was pointed out by Rott (reference 3) for a free or unbounded vortex. The heating in the case of liquids is due to the dissipation. The dissipation is still present in the case of gases of course but it is more than counterbalanced by the cooling due to expansion as the gases approach the low pressure region near the center of the vortex.

In reference 3 only the dissipation due to the tangential velocity is considered in the solution so in the case of liquids it is appropriate to use the approximation (4.18) in comparing results. But Λ_V in that approximation and Λ_G for gases, by (3.6) represent the enthalpy difference divided by V^2, and V has no meaning for Burgers' solution (on which Rott's work is based) since that solution represents an unbounded vortex. Rott presents his results in terms of the enthalpy difference divided by H^* where H^* is defined in terms of the circulation at infinity. This, of course, has no meaning for the present solutions. However, Rott shows that $H^* = 1.22\, v_m^2$ where v_m is the maximum value of v, and this provides a basis for comparison. Accordingly, the quantity $c_p(T_0 - T_w)/(1.22\, v_m^2)$ where T_0 represents the temperature at the origin, has been calculated from the formula

$$\frac{c_p(T_0 - T_w)}{1.22\, v_m^2} = \frac{V^2}{1.22\, v_m^2} \Lambda_G(0)$$

for gases, and

$$\frac{c_p(T_0 - T_w)}{1.22\, v_m^2} = \frac{V^2}{1.22\, v_m^2} \Lambda_V(0)$$

for liquids, where the approximation (4.18) has been assumed. Figure 5.1 shows the variation of this quantity with N_U. Also indicated in the figure are the values obtained by Rott for liquids with infinite Prandtl number and for gases with $Pr = 1/2$, $Pr = 1$, and $Pr = \infty$. These are labeled $N_U = -\infty$ since the vortex considered has no boundary at a finite radius.

The present results seem to approach asymptotic values as $|N_U|$ increases but it is not clear that for gases they approach the values given by Rott. If not, the difference is probably due to the fact that at large Reynolds numbers the solutions of family I-1 are more like the so-called cosine solution than like Burgers' solution. In particular, the axial velocity distribution is more like

$$\frac{w(z,\ r)}{w(z,\ 0)} = \cos\left(\frac{\pi}{2}\frac{r^2}{R^2}\right)$$

than is like

$$\frac{w(z,\ r)}{w(z,\ 0)} = 1$$

(See Part I, p. 50 and p. 64). Thus it is quite likely that the temperature difference approaches something other than Rott's value as N_U approaches minus infinity.

The distributions of Λ_2 shown in Figures 4.1, 4.2, and 4.3 illustrate an interesting effect, namely that at large enough z, by the approximation (4.20), the maximum temperature is not on the centerline $r = 0$ but rather at a radius greater than $R/2$. This behavior is due to the fact that at large z the fluid entering through the porous wall with $w = 0$ must turn rapidly and acquire a substantial axial velocity component. This involves a large contribution to the dissipation near the wall. In fact the very presence of the Λ_2 contribution to (4.17) can be traced to the term

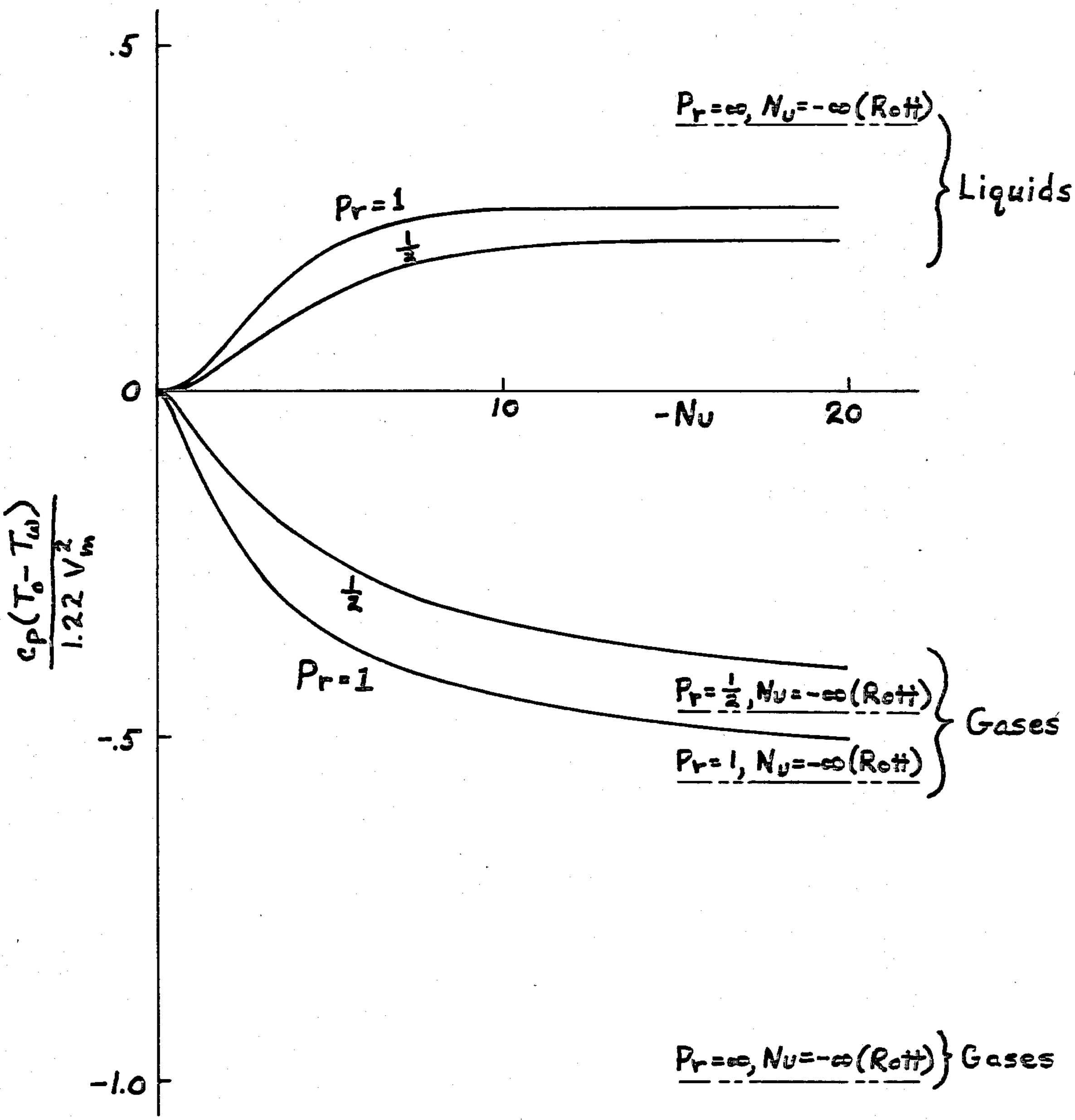

Fig. 5.1 Non-dimensional temperature difference as a function of Reynolds number.

$\mu(\partial w/\partial r)^2$ (which is proportional to z^2) in the dissipation Φ' given by (2.4). The fluid near the centerline is cooler since it entered at a smaller z and thus experienced a milder $\partial w/\partial r$ and less dissipation.

These observations serve as a reminder that the solutions for gases are not valid for large z where the Mach number associated with w becomes appreciable.

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